Comparison of Methods for the Calculation of Rocket Nozzle Wall Temperatures

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Abstract

TEMPERATURE histories and profiles in uncooled rocket nozzle walls can be calculated in various ways. Techniques are available in the literature comprising analytic, numerical, and mixed methods, the latter ones combining analytic and numerical techniques. Four methods are compared for their accuracy and computational speed: an analytic method (AM), a mixed analytic and numerical method (CPM), an analytic approximation based on asymptotic expansions (AEM), and an explicit numerical integration (DIM). In all cases, constant thermal properties are assumed. Temperture histories at the inner wall of the nozzle are given in graphical form and are directly applicable to practical problems. The results are useful for application to similar problems which are also described by the diffusion equation. The choice of which method should be applied for solving similar problems depends, among other things, on the accuracy that can be attained and on the computational speed. All methods that are discussed yield accurate results. The AEM has the greatest computational speed and is recommended in those cases where the expansion is admissible. Otherwise, the AM can be used; the DIM requires most computer time.

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Hot combustion products gradually heat the wall of an uncooled rocket nozzle. Maximum heat transfer is reached near the nozzle throat.[1] In general, the heat flow in the wall is mainly in the outward radial direction, its component in axial direction being negligible. Therefore, the nozzle can be approximated by a number of "isolated" cylindrical rings with dimensions corresponding to the mean dimensions of the corresponding nozzle sections.

For such a cylindrical ring, the equation describing heat transfer in radial direction is[1-3]

$$\frac{\partial\theta}{\partial\bar{t}} - \frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left\{\bar{r}\frac{\partial\theta}{\partial\bar{r}}\right\} = 0 \tag{1}$$

θ is the normalized temperature, $\theta = (T - T_g)/(T_0 - T_g)$. T is the actual temperature, T_g is the temperature of the combustion products, and T_0 is the initial uniform temperature of the ring. The normalized time $\bar{t} = \kappa t/\delta^2$, and the normalized radius $\bar{r} = r/\delta$. The geometry of a ring is given in Fig. 1. The thermal diffusivity κ is assumed to be constant. The appropriate boundary conditions are

$$\bar{t} = 0: \quad \theta = 1 \tag{2a}$$

$$\bar{r} = \bar{r}_0: \quad \partial\theta/\partial\bar{r} = (h_c\delta/k)\cdot\theta \tag{2b}$$

$$\bar{r} = \bar{r}_e: \quad \partial\theta/\partial\bar{r} = 0 \tag{2c}$$

h_c is the convective heat transfer coefficient, and k is the conductivity of the wall. The analytic solution of Eq. (1) together with the boundary conditions of Eqs. (2) is

$$\theta = \sum_{i=1}^{\infty} A_i \exp(-\alpha_i^2\bar{t})\cdot R_i(\bar{r}) \tag{3a}$$

$$R_i(\bar{r}) = J_0(\alpha_i\bar{r}) - Y_0(\alpha_i\bar{r})\cdot J_1(\alpha_i\bar{r}_e)/Y_1(\alpha_i\bar{r}_e) \tag{3b}$$

with the usual notation for Bessel functions.[4] The eigenvalues α_i follow from

$$\{J_1(\alpha_i\bar{r}_0)Y_1(\alpha_i\bar{r}_e) - Y_1(\alpha_i\bar{r}_0)J_1(\alpha_i\bar{r}_e)\}\cdot k\cdot\alpha_i$$
$$= -R_i(\bar{r}_0)\cdot Y_1(\alpha_i\bar{r}_e)\cdot\delta\cdot h_c \tag{4}$$

The numerical solution of Eq. (4) to obtain the subsequent eigenvalues α_i is straightforward.[3] A purely analytic solution of the coefficients A_i containing combinations of Bessel functions is found by standard techniques.[2,3] Results of the analytic solution, AM, are presented in Fig. 2.

The coefficients A_i can also be determined numerically by the application of boundary condition Eq. (2a). Then, for all $\bar{r}$, such that

$$r_0 \le r \le r_e: \quad \sum_{i=1}^{\infty} A_i R_i(\bar{r}) = 1$$

leading to a system of linear equations

$$[A] = [R]^{-1}[I] \tag{5}$$

$[R]^{-1}$ is the inverse of the matrix of the function $R_i(\bar{r}_j)$ of the collocation points r_i, $(i,j = 1,2\ldots N)$. Solutions of Eq. (1) by this collocation point method, CPM, are similar to those obtained by the AM, but owing to the necessity of matrix inversion, the computational speed is reduced considerably.

If the wall is relatively thin, i.e., $\bar{r}_e \gg 1$, the solution of Eq. (1) must approach the flat plate solution. Writing

$$r = (1 - \epsilon y)/\epsilon \tag{6}$$

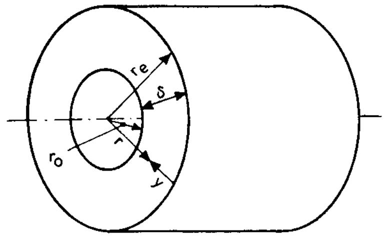

Fig. 1 Geometry and nomenclature of a cylindrical ring representing a nozzle section.

Received June 29, 1978; presented as Paper 78-1044 at the AIAA/SAE 14th Joint Propulsion Conference, Las Vegas, Nev., July 25-27, 1978; synoptic received Dec. 3, 1979. Full paper available from AIAA Library, 555 W. 57th St., New York, N.Y. 10019. Price: Microfiche, $3.00; hard copy, $7.00.

Index categories: Heat Conduction; Solid and Hybrid Rocket Engines.

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Table 1 Computational times for AM, CPM, and AEM

Number of coefficients A_i, N	Computational time, s					
	Coefficients, A_i			Temperature, $\theta(\bar{t},\bar{r})$		
	AM	CPM	AEM	AM	CPM	AEM
10	5.2	18.7	0.6	0.01	0.01	0.01
60	47.5	128.1	3.9	0.02	0.02	0.01

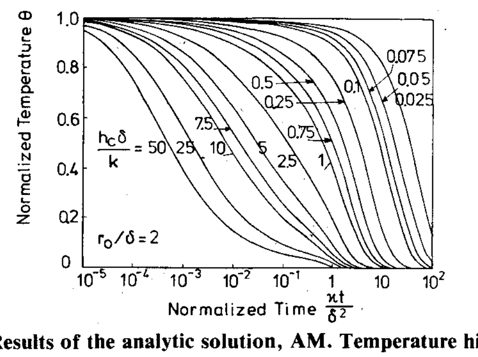

Fig. 2 Results of the analytic solution, AM. Temperature history at the inner wall of the nozzle, $r_0/\delta = 2$.

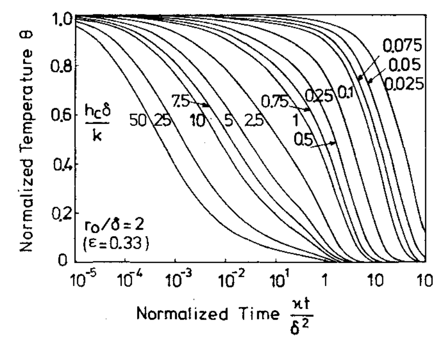

Fig. 3 Results of the asymptotic expansion method, AEM. Temperature history at the inner wall of the nozzle, $r_0/\delta = 2$, $\epsilon = 0.33$.

with $\epsilon = 1/\bar{r}_e$, subsequent substitution of Eq. (6) into the Eqs. (1) and (2), and omitting terms of order ϵ^2 and higher, yields a simple differential equation for the location-dependent part of the temperature

$$\frac{d^2R_i}{dy^2} - \epsilon\frac{dR_i}{dy} + \alpha_i^2 R_i = 0 \tag{7}$$

and for large $\bar{r}_e$, the temperature θ is

$$\theta = \sum_{i=1}^{\infty} A_i \exp\{-\alpha_i^2 t + \epsilon y/2\}\cos(\alpha_i y + \gamma_i)/\cos(\gamma_i) \tag{8}$$

with $\gamma_i = \operatorname{atan}[\epsilon\sqrt{4\alpha_i^2 - \epsilon^2}]$.

Equation (4), like the expression for the coefficient A_i, is replaced accordingly by an equation containing trigonometric functions instead of Bessel functions. If $\epsilon > 2\alpha_i$, the trigonometric functions are replaced by hyperbolic functions. This allows a very rapid computation of the temperature profile and history. Results of this asymptotic expansion method, AEM, are presented in Fig. 3, which should be compared with Fig. 2.

Finally, to compare the three previous methods with a purely numerical method, Eq. (1) is integrated by an explicit finite difference method,[3,5] DIM. Computational speed and accuracy for the explicit and implicit finite difference methods can be assumed to be comparable, although for large times the implicit method may require fewer computations.[5]

An impression of the accuracy is obtained by comparing the results. For the CPM and the DIM, the difference with the AM is less than 10^{-2}. The difference between the AEM and AM is shown in Fig. 4 as an example. As the accuracy that can be attained with all methods is quite good, accuracy is not considered decisive for the choice of a particular method. Large relative errors, which can occur for large $\bar{t}$, are due to the very small values of θ; then the absolute errors are extremely small. For the AEM, it is found that even if ϵ is large, i.e., $\epsilon = 0.5$, the accuracy of the method is still good. Table 1 shows the relative computational speed of the AM, CPM, and AEM. The coefficients A_i have to be calculated only once together with the eigenvalues α_i. The time to calculate the first N coefficients A_i and N eigenvalues α_i is listed. The time to calculate $\theta(\bar{t},\bar{r})$ once is also shown.

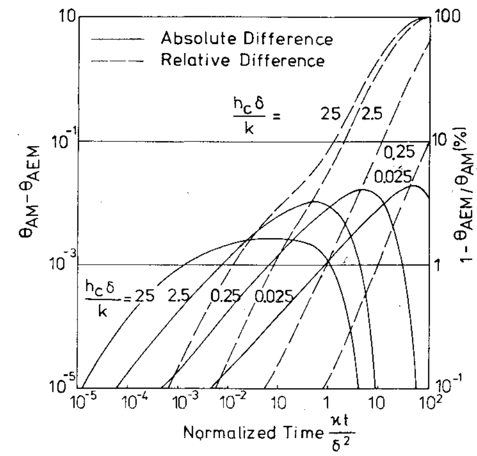

Fig. 4 Comparison of the results obtained by the AM and AEM, $r_0/\delta = 2$, $\epsilon = 0.33$.

Due to the use of simple trigonometric functions, the AEM is most efficient. The time for a numerical integration may be compared with the previous methods. The DIM requires 0.35 ms per step. With a time increment $\Delta\bar{t} = 10^{-3}$ and a radial increment $\Delta\bar{r} = 1/160$, about 56 s are required to obtain the temperature distributions at $\bar{t} = 1$, and for $\bar{t} = 100$, about 9 min of computer time are required. All computations have been made on the IBM 360/158 of Delft University of Technology.

If no suitable tabular or graphical[1,3,6] solutions of the diffusion equation are available, a solution based on asymptotic expansions is recommended in those cases where ϵ is sufficiently small. For the problem discussed here, $\epsilon = 0.5$ is still quite acceptable. For larger values of ϵ, an analytic solution, AM, will be most efficient.

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